Nicholas Hender

Area Of Study 1, Motion In One And Two Dimensions, Study Notes 7

Circular Motion in a Vertical Plane

For an object moving circularly in a vertical plane within proximity to the Earth's surface, the overall centripetal force (F_C) that accelerates the object in its circular path is affected by the weight force (F_W) of the object's mass (due to gravity).

F_C must equal the sum of all forces acting, ΣF, as long as an object remains moving in a circular path.

F_C is always directed towards the circle's centre.

Forces contributing to F_C are:
- F_W, and
- F_N, the normal reaction force "experienced" by the object as a result of F_W and F_C.

Therefore:

$$\boxed{\Sigma\vec{F} = \vec{F}_C = \vec{F}_N + \vec{F}_W}$$ (vectorially, regardless of position in circle)

As with circular motion in a horizontal plane, F_C in a vertical plane is given by the formula $\boldsymbol{mv^2/r}$.

Also, in keeping with Newton's Second Law of Motion, F_W is given by the formula $\boldsymbol{mg}$.

So for magnitude, without needing to deal with vector addition and subtraction:

At **bottom** of vertical circle: $F_N = F_C + F_W$ (in terms only of magnitude, as F_C is upwards and F_W is downwards)

$= \frac{mv^2}{r} + mg$ (a positive result indicating an **upwards** direction)

At **top** of vertical circle: $F_N = F_W - F_C$ (in terms only of magnitude, as F_C and F_W are both downwards)

$= mg - \frac{mv^2}{r}$ (a negative result indicating a **downwards** direction)

Where ΣF = Force (overall, keeping the object in circular motion, equal to F_C), in N
F_C = Force (centripetal, always towards the circle's centre, varying with v and r), in N
F_W = Force (weight, always downwards, constant with m and g), in N
F_N = Force (normal reaction, the "apparent" force experienced by the object, varying with F_C), in N
m = mass (of object in circular motion), in kg
v = magnitude of velocity (speed), in ms^{-1}
g = acceleration due to gravity, as 9.8 or 10 ms^{-2}
r = radius (of circle), in m

F_N, being the force "experienced" by the object in circular motion can be thought of as the magnitude and direction of the push you would feel if undergoing the circular motion yourself, in an aeroplane or roller coaster, for example.

F_N is equal to what's also referred to as "apparent weight"; the amount of weight you perceive yourself to have due to the reaction force against it. It is also what's sometimes referred to as "centrifugal" force (as opposed to centripetal), but use of term is best avoided due it being commonly misunderstood.

Examples of circular motion in a vertical plane are:

- A roller coaster that traverses a vertical loop.
- An aeroplane performing a "loop-the-loop" manoeuvre.
- Motorcycle stunt riders in a "globe-of-death".
- A skateboarder in a "full-pipe".
- The yo-yo trick "around the world".
- Swinging a bucket containing water over your head fast enough such that the water stays in the bucket.

In each of these examples, the object remains in circular motion as long as the **speed at the top of the loop** is such that **the magnitude of F_C is equal to or greater than that of F_W**.

For an unattached object traversing the outside of a vertical loop, such as a car driving over a hump, the object stays in circular motion as long as the **speed at the top of the loop** is such that **F_C is equal to or less than F_W**.

For an object traversing the **inside** of a vertical loop:
the object falls from the loop if the magnitude of F_C is less than that of F_W.

For an unattached object traversing the **outside** of a vertical loop, such as a car driving over a hump:
the object departs from the loop if the magnitude of F_C is greater than that of F_W.
So in the case of the car driving over a hump, it would become airborne.

To find the speed required at the top of a loop to maintain circular motion in a vertical plane:

$$F_C = F_W$$

$$\rightarrow \frac{mv^2}{r} = mg$$ .... substituting formulæ for F_C and F_W

$$\rightarrow \frac{v^2}{r} = g$$ .... m cancels out

$$\boxed{\rightarrow v = \sqrt{gr}}$$ .... transposed

Where v = magnitude of velocity (speed) required to maintain vertical circle, in ms^{-1}
g = acceleration due to gravity, as 9.8 or 10 ms^{-2}
r = radius (of circle), in m

If g is constant (which it is if proximity to the Earth's surface is maintained), this formula shows that v is independent of mass, and only affected by r.